

On the Exceptional Case in a Characterization of the Arcs of a Complete Graph

Abstract: It is known that a certain simple set of properties characterize the relationship of adjacency of the $n(n-1)/2$ arcs of the complete graph of order n , when $n \neq 8$, and that these properties are not a sufficient characterization when $n = 8$ (see [1], [2], [3], [4], and [7]). The present paper describes a method for enumerating all counter-examples when $n = 8$.

1. Introduction

In an abstract set S of $n(n-1)/2$ objects, let there be defined two symmetric relationships, "first associates" and "second associates" on pairs of distinct objects in S , so that any two distinct objects are either first associates or second associates. Suppose that

every object has $2(n-2)$ first associates; (1.1)

if two objects are first associates, there are exactly $n-2$ other objects, each of which is a first associate of each of the two; (1.2)

if two objects are second associates, there are exactly four objects, each of which is a first associate of each of the two. (1.3)

It is known that, provided $n \neq 8$, properties (1.1)-(1.3) suffice to prove that S is isomorphic to the set of $n(n-1)/2$ arcs of the complete graph determined by n points, first associates corresponding to a pair of arcs with a common end point, second associates corresponding to a pair of arcs without an end point. This does not necessarily hold, however, when $n = 8$ ([1], [2], [4]). A complete enumeration of all counter-examples when $n = 8$ has also been given (see [2]).

The purpose of the present manuscript is to indicate an alternate method of arriving at the enumeration of these counter-examples, a method based on

the exploitation of properties of the eigenvalues and eigenvectors of symmetric matrices and their principal submatrices. As will be seen, the calculations become rather formidable, and have therefore not been carried to completion. In view of the results of [2], there would seem to be no point in carrying them further.

What is given in the sequel can best be described as a *method* which, when properly executed, will complete the discussion of all models of the postulate system (1.1)-(1.3), a problem first posed by Connor and solved by him [3] for $n \geq 9$. The method will also furnish evidence to supplant that offered in [4] and [5] of the usefulness of eigenvalue and eigenvector considerations in combinatorial problems, a development predicted in [6].

Now to begin the attack on the problem.

Number the elements of S from 1 to 28 in any order. Let A be a square matrix of order 28 defined by

$$A = (a_{ij})$$

$$= \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i \text{ and } j \text{ are first associates} \\ 0 & \text{if } i \text{ and } j \text{ are second associates} \end{cases} \quad (1.4)$$

Then it is obvious that (1.1)-(1.3) with $n = 8$ imply $A^2 = 8I + 2A + 4J$, where I is the identity matrix and J is the matrix every entry of which is 1. It is shown in [2] that this equation implies that the eigenvalues of A are

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- (a) 12, with multiplicity 1, $(1, 1, \dots, 1)$
corresponding eigenvector
(b) 4, with multiplicity 7, and
(c) -2, with multiplicity 20. (1.5)

We shall now make use of some well-known properties of (real) symmetric matrices, which we list now for ready reference:

Let M be a real symmetric matrix of order m , with α as maximum eigenvalue, and $\beta < \alpha$ the minimum eigenvalue. Let b be the multiplicity of β . Let K be a submatrix of order k , with δ the minimum eigenvalue of K , y a corresponding eigenvector.

Then $\delta \geq \beta$, and if $\delta = \beta$, y is orthogonal to that part of every column of M not meeting K consisting of those rows that do meet K . (1.6)

If $k + b > m$, $\delta = \beta$. (1.7)

Using other properties of eigenvalues and eigenvectors, it was shown in [4] that if

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is a principal submatrix of A , consisting (say) of the first three rows and columns of A , then there is either exactly one (say object 4) which is a second associate of 1, 2, and 3, or exactly one object (say object 4) which is a first associate of 1, 2, 3; and that, in the former case, the first four rows of A may be taken to be

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

(1.8)(a)

In the latter case, the first four rows of A may be taken to be

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

(1.8)(b)

2. Construction of examples

Let us tentatively assume that A contains

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \quad (2.1)$$

as a principal submatrix, say from rows and columns 1-6. It is an easy consequence of (1.1) and (1.3) that there are exactly six objects, say 7-12, which are second associates of both 1 and 2.

We first show that no more than two of these objects can be second associates of all of 3, 4, 5, 6. Suppose as many as three of them (say 7, 8, 9) were. Then (2.1) would be imbedded in a principal submatrix of order 9

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{B} & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & \boxed{B} & \\ 0 & 0 & 0 & 0 & 0 & 0 & & & \boxed{B} \end{bmatrix}, \quad (2.2)$$

where B is a symmetric $(0, 1)$ matrix of order 3, with 0 on the main diagonal. Such a matrix cannot have -2 as an eigenvalue. Since (2.2) is the direct sum of (2.1) and B , and neither (2.1) nor B has -2 as an eigenvalue, it follows that -2 is not an eigenvalue of (2.2). But, in view of (1.5) (c), this would violate (1.7).

We know, therefore, that each of at least four objects of 7-12 has the property of being a first associate of at least one of 3, 4, 5, 6. Assume, say, that 7 is a first associate of 3. Since 1, 2, 7 are mutually second associates, and 3 a first associate of 1, 2, 7, it follows we are in situation (1.8) (b), and 7 is a second associate of 4, 5, 6. Again, by (1.8) (b), 3 is a second associate of 8-12, and 7 is a first associate of 8-12. This line of reasoning shows that, after some renumbering, one may assume that the principal submatrix of A formed by rows and columns 1-12 is

$$\begin{pmatrix}
 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & x \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & x & 0
 \end{pmatrix}. \quad (2.3)$$

Now 1, 2, 11 are mutually second associates; it is clear from (2.3) that no object is a first associate of all of 1, 2, 11, so we must by (1.8) (a) have an object which is a second associate of 1, 2, and 11. The only possibility is 12. Hence, in (2.3) the quantity $x = 0$, and we rewrite

$$\begin{pmatrix}
 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0
 \end{pmatrix}. \quad (2.3)$$

Each of rows 3-10 contains exactly six ones in (2.3), so by (1.1) the remainder of each of these rows in the sixteen remaining columns contains exactly six 1's. By (1.2), it follows that in columns 13-28, rows 3 and 7 are identical, rows 4 and 8 are identical, rows 5 and 9 are identical, and rows 6 and 10 are identical. Furthermore, by (1.2), (1.3), and (2.3), any two rows not "paired" (i.e., not 3 and 7, 4 and 8, 5 and 9, or 6 and 10) in 3-10 have exactly two common ones in columns 13-28. Also, by (1.1)-(1.3), four of columns 13-28 contain ones in rows 1 and 11, zeroes in rows 2 and 12; four

contain ones in rows 1 and 12, zeroes in rows 2 and 11; four contain ones in rows 2 and 11, zeroes in rows 1 and 12; and four contain ones in rows 2 and 12, zeroes in rows 1 and 11.

For simplicity, let us now look at the submatrix of order 8 of (2.3) obtained from rows and columns 1-4 and 9-12. We get

$$\begin{pmatrix}
 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0
 \end{pmatrix}. \quad (2.4)$$

(We shall continue to call these rows and columns 1, 2, 3, 4, 9, 10, 11, 12). It is clear from the foregoing paragraph that when one describes the entries of any one of columns 13-18 in the rows 1-4 and 9-12, one also automatically describes the entries in rows 5-8.

If we note that (2.4) is invariant with respect to:

interchanges of 1 and 2,
interchanges of 11 and 12,
interchanges of 3 and 4,
interchanges of 9 and 10,

and the mapping

$$\begin{aligned}
 1 &\leftrightarrow 12 \\
 2 &\leftrightarrow 11 \\
 3 &\leftrightarrow 10 \\
 4 &\leftrightarrow 9,
 \end{aligned}$$

then any of the possible columns 13-28, rows 1-4 and 9-12, is represented after permutations as exactly one of the following:

$$\begin{matrix}
 \text{(a)} & \text{(b)} & \text{(c)} & \text{(d)} & \text{(e)} & \text{(f)} \\
 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}
 \end{matrix}. \quad (2.5)$$

Now consider the matrices of order 9 obtained by appending each of the six columns of (2.5) to (2.4), the corresponding row, and 0 on the main diagonal. In each case, -2 is an eigenvalue of the matrix of order 9, and here are the corresponding eigenvectors of order 9:

$$\begin{array}{cccccc}
 \text{(a)} & \text{(b)} & \text{(c)} & \text{(d)} & \text{(e)} & \text{(f)} \\
 \begin{bmatrix} 2 \\ 1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 2 \\ -2 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ -1 \\ 1 \\ 2 \\ -2 \end{bmatrix} & \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \\ -1 \\ 1 \\ 2 \\ -2 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ -1 \\ 0 \\ 1 \\ -2 \end{bmatrix} & \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ -2 \end{bmatrix} & \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \\ 1 \\ -1 \\ 0 \\ -2 \end{bmatrix}
 \end{array} \quad (2.6)$$

Notice that in each case, every entry in rows 3, 4, 9, 10 is ± 1 ; the entry corresponding to the added row is -2 . Also, in cases (2.5) (a), (c), (d), and (f), when the number of ones in rows 3, 4, 9, 10 is even, the sum of the entries in 1 and 12 in (2.6) is even, and so is the sum of the entries in 2 and 11, whereas the sum of the entries in 1 and 11 is odd, and so is the sum of the entries in 2 and 12. In cases (b) and (e), where the number of ones in rows 3, 4, 9, 10 is odd, the situation is reversed. In view of (1.6), these facts imply that eight of the columns 13-28 contain an even number of ones in rows, 3, 4, 9, 10, and the remaining eight columns contain an odd number of ones in these rows. (2.7)

What possibilities are there for the submatrix B of A formed by rows 3, 4, 9, 10 and columns 13-28? Recall that each row contains six 1's and the product of any two rows of B is 2. We will consider all possibilities in lexicographic order, ranking in terms of largest number of columns with largest number of ones.

We cannot have two columns with four 1's, since among other arguments there would then be sixteen additional columns, each containing exactly one 1, and there are only sixteen columns in B . We cannot have one column with four 1's, and two or more columns with three 1's, since this would yield two or more rows with an inner product of 3. Could we have exactly one column with four

1's, and exactly one column with three 1's? Then we would have

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

which has four columns with an even number of 1's, twelve columns with an odd number of 1's. Thus this must be ruled out.

Could we have exactly one column with four 1's, and no columns with three 1's? Then there will be eight columns with exactly one 1. Then B may be taken to be

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \quad (2.8)$$

This is true since if we had a column which appears in rows 1-4 and 9-12 as

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

then we cannot have another column identical to the above in those rows, for it would make an inner product of 4 or 2 with (2.6)(b), violating (1.6). We cannot have another column

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

in those rows, for either would make an inner pro-

duct different from 0 with (2.6)(b), also violating (1.6). So there are at most two columns which, in B , have exactly one 1 and that 1 in the same row. From these considerations, the appearance of B in (2.8) follows.

This completes the discussion of possibilities for B in which there are one or more columns with four 1's.

Suppose that the largest number of ones in a column of B is 3. Suppose there are four of them. Then B would have the appearance

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

with every column containing an odd number of 1's, which is impossible.

Suppose there are exactly three columns containing three 1's. Then B would have the appearance

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

which has thirteen columns containing an odd number of 1's and is impossible.

Suppose B contains exactly two columns with three 1's. These three 1's may all be in the same rows or not. If they are in the same rows, then B would have the appearance

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}. \quad (2.9)$$

If they are not in the same rows, then B would have the appearance

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}. \quad (2.10)$$

Now suppose B contains exactly one column with three 1's. Then B would be

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where twelve columns have an even number of 1's, which is impossible.

If there are no columns containing as many as three 1's, then, for the reasons explained in deriving (2.8), we may assume B begins

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

and it is easy to show that at least one of the remaining columns must have more than two 1's, which case has been excluded.

Thus our examination has produced three possible cases, namely (2.8), (2.9), and (2.10) for B . Notice that the allowable renumbering of 3-10 discussed earlier might produce other cases, but these would be, of course, equivalent to the three cases considered.

Let us now see how each of these three possibilities for B can be imbedded in a matrix C formed by rows 1-4 and 9-12, and columns 13-28. Consider first (2.8). There is no loss of generality in assuming that the first column of the corresponding C is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then examination of (2.6)(a) together with (1.6) implies that the second column of C is

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

The facts established in (2.7) show that it is no loss of generality to assume the next 8 columns of C are

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

All that remains is to consider the last six columns of C . The last four of those columns have a 1 in rows 3 or 4, and a 1 in either row 9 and 10. Examination of (2.6)(c) shows that, if the column of C containing column

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

of (2.8) is

$$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

then, by (1.6), the last four columns of C would each have a 1 in rows 1 and 12. Since the first two columns of C also have a 1 in rows 1 and 12, we would have a violation of the last sentence of the paragraph following (2.3).

Hence, the matrix C containing (2.8) is

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Bearing in mind that, in columns 13-28, the above automatically describes rows 5-8, along with 1-4 and 9-12, we see that we have now fully specified one way (arising from (2.8)) that all rows and columns containing (2.3) can be completed in the matrix A . To determine the remainder of the matrix A , all we need do is invoke (2.6). This produces the matrix of order 28:

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \quad (2.11)$$

The matrix (2.11) will provide a counter-example if, and only if, the consequences of (1.4) and (1.1)-(1.3) hold with $n = 8$. This is indeed the case.

Arguments similar to the above yield, from (2.9) and (2.10) respectively, matrices (2.12) and (2.13), displayed on page opposite.

It has been calculated that (2.12) is indeed a counter-example. The calculation has not been performed for (2.13).

It is possible that a suitable renumbering may show that (2.11)-(2.13) are not all distinct. This has not been checked.

3. Further possible counter-examples

For the remainder of this discussion, we may assume A contains no principal submatrix of the form (2.1). Consider now the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}. \quad (3.1)$$

As was shown in [4], if (3.1) is not a principal

$$\begin{pmatrix}
0011110000001010101010101010 \\
0011110000000101010101010101 \\
11011110000001100001111000000 \\
1110110100000011001100110000 \\
1111010010000000111100001100 \\
1111100001000000000011111100 \\
001000011111100001111000000 \\
0001001011110011001100110000 \\
0000101101110000111100001100 \\
0000011110110000000011111100 \\
0000001111000101010110101010 \\
0000001111001010101001010101 \\
1010001000010110101011000011 \\
0110001000101001010111000011 \\
1001000100011001101000110011 \\
0101000100100110010100110011 \\
1000100010011010011000001111 \\
0100100010100101100100001111 \\
1011101110011010100100000000 \\
0111101110100101011000000000 \\
1010011001101100000001101010 \\
0110011001011100000010010101 \\
1001010101100011000010011010 \\
0101010101010011000001100101 \\
1000110011100000110010100110 \\
0100110011010000110001011001 \\
10000000010111110010101001 \\
0100000000011111110001010110
\end{pmatrix} \quad (2.12)$$

$$\begin{pmatrix}
0011110000001010101001101001 \\
0011110000000101010110010110 \\
1101111000000010110010001010 \\
1110110100000001110001000101 \\
1111010010000000001111000011 \\
1111100001000000000000111111 \\
0010000111110010110010001010 \\
0001001011110001110001000101 \\
0000101101110000001111000011 \\
0000011110110000000000111111 \\
0000001111000110011010100101 \\
0000001111001001100101011010 \\
1000000000010111101101111000 \\
0100000000101011011110110100 \\
1010001000101100111010101000 \\
0101000100011100110101010100 \\
1011001100011011010001001000 \\
0111001100100111100010000100 \\
1000100010101110000111100001 \\
0100100010011101001011010010 \\
0110101010100110011100000010 \\
1001100110011001101100000001 \\
1000010001101110001000011101 \\
0100010001011101000100101110 \\
1010011001011010100000110010 \\
0101010101100101010000110001 \\
0110111011010000000110011000 \\
1001110111100000001001100100
\end{pmatrix} \quad (2.13)$$

submatrix of A , then S is isomorphic to the arcs of the complete graph on n points:

Suppose (3.1) is a principal submatrix of A . Then, by (1.8)(b), A contains a principal submatrix of order 9 of the form

$$\begin{pmatrix}
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & x & x & x & x & x \\
1 & 0 & 0 & 0 & x & 0 & x & x & x \\
1 & 0 & 0 & 0 & x & x & 0 & x & x \\
1 & 0 & 0 & 0 & x & x & x & 0 & x \\
1 & 0 & 0 & 0 & x & x & x & x & 0
\end{pmatrix}. \quad (3.2)$$

If every $x = 1$, then -2 is not an eigenvalue of (3.2), which violates (1.7). So at least one $x = 0$, say $a_{56} = a_{65} = 0$. It follows that the first six rows and columns of (3.2) form

$$\begin{pmatrix}
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \quad (3.3)$$

which has -2 as its least eigenvalue, and $(2, 1, 1, -2, -1, -1)$ as corresponding eigenvector. By (1.6), it follows that, in (3.2),

$$\begin{aligned}
a_{57} &= a_{58} = a_{59} = a_{67} = a_{68} = a_{69} \\
&= a_{75} = a_{85} = a_{95} = a_{76} = a_{86} = a_{96} = 1,
\end{aligned}$$

so we may rewrite

$$\begin{pmatrix}
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & x & x \\
1 & 0 & 0 & 0 & 1 & 1 & x & 0 & x \\
1 & 0 & 0 & 0 & 1 & 1 & x & x & 0
\end{pmatrix}. \quad (3.2)$$

Thus, 5 and 6 are second associates, 1, 7, 8, 9 are first associates of 5 and 6, 1 is a first associate of 7, 8, 9. It follows from the stipulation beginning this section that at least two of 7, 8, 9 are second associates. Without loss of generality, assume 7 and 8 are second associates; i.e., in (3.2),

$$a_{78} = a_{87} = 0$$

Then rows and columns 5, 6, 7, 8 of (3.2) form a principal submatrix

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \quad (3.4)$$

of A , with -2 as eigenvalue, $(1, 1, -1, -1)$ corresponding eigenvector. Application of (1.6) then implies that, in (3.2), $a_{79} = a_{89} = a_{98} = a_{97} = 1$. So we have finally concluded that we may write (3.2) as

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \quad (3.2)$$

Referring back to (1.8)(b), and invoking (3.2), the supposition that A contains (3.1) as a principal submatrix implies that A contains

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & x & x & x \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & x & x & x \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & x & x & x \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & x & x & x \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & x & x & x \\ 0 & 1 & 1 & 1 & x & x & x & x & 0 & x & x & x \\ 0 & 1 & 1 & 1 & x & x & x & x & 0 & x & x & x \\ 0 & 1 & 1 & 1 & x & x & x & x & 0 & x & x & x \end{pmatrix} \quad (3.5)$$

The argument used before in connection with (3.3) show that, in (3.8),

$$\begin{aligned} a_{5,10} &= a_{5,11} = a_{5,12} = a_{6,10} = a_{6,11} \\ &= a_{6,12} = a_{10,5} = a_{11,5} = a_{12,5} \\ &= a_{10,6} = a_{11,6} = a_{12,6} = 0, \end{aligned} \quad (3.6)$$

and considerations applied to (3.4), in view of (3.6),

$$\begin{aligned} \text{imply } a_{7,10} &= a_{7,11} = a_{7,12} = a_{8,10} = a_{8,11} \\ &= a_{8,12} = a_{10,7} = a_{11,7} = a_{12,7} \\ &= a_{10,8} = a_{11,8} = a_{12,8} = 0. \end{aligned}$$

Hence, we may assume (3.5) has the appearance

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & x & x & x \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & x & 0 & x & x \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & x & x & 0 & x \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & x & x & x & 0 \end{pmatrix} \quad (3.5)$$

Now 2, 3, 9 are mutually second associates. 1, 5, 6, 7, 8 are first associates of 9, second associates of 2 and 3. It follows from (1.8)(b) that exactly one of the three x 's in column 9 of (3.5) is 1, say $a_{12,9} = a_{9,12} = 1$, while $a_{9,10} = a_{9,11} = a_{11,9} = a_{10,9} = 0$. Further, since 7, 8, 12 are mutually second associates, and 9 is now known to be a first associate of each, we infer from (1.8)(b) that $a_{10,12} = a_{11,12} = a_{12,10} = a_{12,11} = 1$. Further, 7, 8, 11 are mutually second associates. There is no object which is a first associate of all three, so by (1.8)(a), $a_{10,11} = a_{11,10} = 0$. Hence, we know we can write (3.5) as

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \quad (3.5)$$

Let us now look at the submatrix of (3.5) formed by rows and columns 2, 3, 11, 12, 1, 7, 5, 6, and rewrite it so that 2 is first, 3 second, 11 third, et cetera. We obtain

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}, \quad (3.7)$$

a matrix which looks identical to (2.4), and suggests that the technique used in Section 3 may be applied here, albeit with slightly different side conditions.

First, if we know how any column other than 1-12 "meets" rows 2, 3, 11, 12, 1, 7, 5, 6, we know how it meets rows 4, 8, 9, 10. For it is clear from (3.5) and (1.1)-(1.3) that the entries in rows 1 and 4 will be identical; the entries in 7 and 8 completely different; 12 and 9 identical; and 11 and 10 completely different.

Second, if we let B be the submatrix of A formed by rows 11, 12, 1, 7 (written in that order) and columns 13-28, then (3.5) and (1.1)-(1.3) show that

$$BB^T = \begin{pmatrix} 8 & 3 & 3 & 4 \\ 3 & 6 & 2 & 3 \\ 3 & 2 & 6 & 3 \\ 4 & 3 & 3 & 8 \end{pmatrix}. \quad (3.8)$$

Third, in 13-28, there are

- four columns with 1's in rows 2 and 5,
and 0's in rows 3 and 6;
- four columns with 1's in rows 2 and 6,
and 0's in rows 3 and 5;
- four columns with 1's in rows 3 and 5,
and 0's in rows 2 and 6;
- four columns with 1's in rows 3 and 6,
and 0's in rows 2 and 5.

The material in (2.5) and (2.6), and the remarks of (2.7) and the subsequent paragraph, apply *mutatis mutandis* to the current situation.

In particular, an examination of all possibilities for B satisfying the above facts and (3.8) yields these four possibilities (and no others):

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad (3.9)$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}, \quad (3.10)$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}, \quad (3.11)$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}. \quad (3.12)$$

A development of the full matrix A containing (3.9) obtained by using (2.5) and (2.6) is

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \end{pmatrix} \quad (3.13)$$

after we have effected the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 5 & 1 & 2 & 9 & 7 & 8 & 6 & 10 & 11 & 12 & 3 & 4 \end{pmatrix}.$$

Note that the outlined box, together with the third row and column, form a principal submatrix

of (3.13) of the form (2.1), so (3.13) *either* does not satisfy (1.1)-(1.3), *or* it is a counter-example met in Section 3. In either case, it need not be considered further.

The "immersion" of (3.10)-(3.12) respectively in matrices A has not been calculated.

In any case, assuming all the calculations are correct, the task that remains after such immersions have been deduced from (2.5) and (2.6) is to test if they satisfy the postulates, and then to test among those that do, how many are different.

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